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THE DIAGNOSTIC PROBLEM FOR ELASTIC SEMIBOUNDED BODIES*

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By a diagnostic problem we mean the problem of determining the characteristics of a material from information which has been obtained experimentally on the physical fields which arise in a body due to the action of external actions which have been chosen in a special way /1/. In this paper we investigate the problem of determining the density and the rigidity characteristics of a weakly inhomogeneous and anisotropic linearly elastic medium from measurements on the surface of the body of the parameters of the elastic processes taking place in it which corresponds to the possibilities of modern measuring techniques /2/. We confine ourselves to the treatment of almost stationary processes, that is, of processes which are initiated by loads which give rise to stationary processes of the same form in a reference homogeneous and isotropic body. Hence, a certain lack of stationary behaviour of the investigated processes in the body being studied is assumed to be exclusively associated with the weak inhomogeneity and anisotropy of this body. The semibounded body with a weak curvilinear boundary which is considered may serve as a model of a massive component. In the mathematical scheme, the problem being investigated belongs to the class of inverse problems in mathematical physics /3/.

1. The propagation of elastic waves in an inhomogeneous anisotropic medium which occupies a semibounded domain $\Omega=\{-\infty<x_1, x_2<\infty,\ \gamma(x_1,x_2)\leqslant x_3<\infty,\ 0\leqslant\gamma\leqslant M-\text{const},\ \gamma\in C^1(R^2)\}$ (Fig.1) is described by the equations /4/

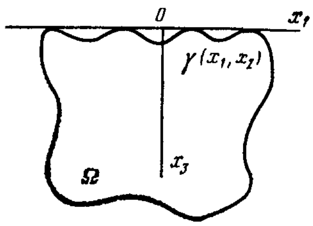

Fig.1

$$\rho u_i^{\cdot\cdot}-(C_{ijkl}u_{k,l})_{,j}=f_i;\quad u_i\in C^4(\Omega\times R_+),\quad \rho, C_{ijkl}\in C^1(\Omega) \tag{1.1}$$

which are closed with the initial and boundary conditions

$$u_i|_{t=0}=\varphi_i(\mathbf{x}),\quad u_i^{\cdot}|_{t=0}=\psi_i(\mathbf{x}) \tag{1.2}$$

$$(C_{ijkl}u_{k,l}n_j)|_{x_3=\gamma}=p_i(x_1,x_2,t) \tag{1.3}$$

$$n_r=\gamma_{,r}(1+\gamma_{,1}+\gamma_{,2})^{-1/2},\quad n_3=(1+\gamma_{,1}+\gamma_{,2})^{-1/2}$$

The density ρ and the components of the elasticity tensor C_{ijkl} depend on the spatial variables $\mathbf{x}=(x_1,x_2,x_3)$, while the components of the displacement vector $\mathbf{u}=(u_1,u_2,u_3)$ are functions of $\mathbf{x}$ and the t. The dots denote time derivatives while the index after a comma denotes a derivative with respect to the corresponding coordinate. Unless otherwise stated summation is carried out over a repeated index and, everywhere, $i,j,k,l=1,2,3$; $r=1,2$; $n=1,2,\ldots,N$.

The diagnostic problem under consideration involves the determination of $\rho(\mathbf{x})$ and $C_{ijkl}(\mathbf{x})$ from several problems of the form of (1.1)-(1.3) under N different types of loads (after putting $u_i^n\to u_i$, $\{\varphi,\psi,p,f\}_i^n\to\{\varphi,\psi,p,f\}_i$) using the ancillary information

$$u_i^n|_{x_3=\gamma}=\chi_i^n(x_1,x_2,t),\quad C_{ijkl}|_{x_3=\gamma}=C_{ijkl}^{(0)} \tag{1.4}$$

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which is assumed to have been obtained as the result of direct measurements. The number N, which corresponds to the number of experiments with different types of loads depends on the type of anisotropy in the medium under investigation (the number of functions required).

We shall subsequently assume that the medium investigated is weakly inhomogeneous and anisotropic, that is, the quantities $|\rho - \rho^{(0)}|/\rho^{(0)}$, $\| C_{ijkl} - C_{ijkl}^{(0)} \|_{C(\Omega)}/\min(\lambda^{(0)}, \mu^{(0)})$ are of an order of smallness ε $(0 < \varepsilon \ll 1)$, where $\rho^{(0)}$ and $C_{ijkl}^{(0)}$ are constant characteristics which correspond to a certain homogeneous isotropic medium and this means that $C_{ijkl}^{(0)} = \lambda^{(0)} \delta_{ij}\delta_{kl} + \mu^{(0)} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ /4/, $0 < \lambda^{(0)}, \mu^{(0)}$ are Lamé coefficients and δ_{ij} is the Kronecker delta.

We note that a weakly inhomogeneity and anisotropy in the elastic properties of the material does not at all mean that it will also be so in the strength scheme. For instance, the α-irradiation of metals changes their Young's modulus by 10-15% (Poisson's ratio varies to a lesser extent) while the yield stress increases several-fold /5/. It follows from this that non-uniform irradiation causes a weak inhomogeneity in the elastic properties and a strong inhomogeneity in the strength properties of metals. Hence, by determining the elastic inhomogeneity of a material within the framework of a diagnostic problem and separating out the zones with a different level of radiation defects, it is possible to estimate the reserves of strength of a component.

Let us compare an elastic process $u_i^n(\mathbf{x}, t)$ with a process $u_i^{(0)n}(\mathbf{x}, t)$ which is initiated in an analogous manner and takes place in a domain Ω which is occupied by a reference homogeneous and isotropic medium. $u_i^{(0)n}$ is described by Eqs.(1.1)-(1.3) after making the replacement $\{\rho, C_{ijkl}\}^{(0)} \to \{\rho, C_{ijkl}\}(\mathbf{x})$. We shall assume that the effect of the anisotropy and inhomogeneity of the medium investigated on the quantitative characteristics of the processes excited in it is rather small and, on the surface of the body,

$$u_i^n|_{x_3=\gamma} = u_i^{(0)n}|_{x_3=\gamma} + \varepsilon u_i^{(1)n}|_{x_3=\gamma}, \quad 0 < \varepsilon \ll 1$$

and

$$\| u_i^{(1)n}|_{x_3=\gamma} \|_{C^2} = O(\| u_i^{(0)n}|_{x_3=\gamma} \|_{C^2})$$

Let us also assume that the required characteristics of the medium investigated and the elastic processes occurring in it are analytical with respect to the small parameter ε which has been introduced, that is,

$$\{\rho, C_{ijkl}, u_i^n\} = \sum_{m=0}^{\infty} \varepsilon^m \{\rho, C_{ijkl}, u_i^n\}^{(m)}$$

We note that, having confined ourselves to the treatment of bodies with a weakly curvilinear boundary, that is, by putting $\gamma(x_1, x_2) = \varepsilon\alpha(x_1, x_2)$, it is possible with the aid of Taylor's series expansion to pass from the values on the surface $x_3 = \gamma(x_1, x_2)$ to the corresponding values on the hyperplane $x_3 = 0$. The assumptions which have been adopted enable one, in accordance with the procedure used in the perturbation method, to pass from relationships (1.1)-(1.4) to the chain of equations

$$\rho^{(0)} u_i^{(0)n\cdot\cdot} - C_{ijkl}^{(0)} u_{k,lj}^{(0)n} = f_i, \ldots, \rho^{(0)} u_i^{(m)n\cdot\cdot} - C_{ijkl}^{(0)} u_{k,lj}^{(m)n} = \sum_{s=0}^{m-1} ((C_{ijkl}^{(m-s)} u_{k,l}^{(s)n})_{,j} - \rho^{(m-s)} u_i^{(s)n\cdot\cdot}), \ldots \tag{1.5}$$

which are closed by the initial and boundary value equations

$$u_i^{(m)n}|_{t=0} = \delta_{0m}\varphi_i, \quad u_i^{(m)n\cdot}|_{t=0} = \delta_{0m}\psi_i^n \tag{1.6}$$

$$\left\{ \sum_{s=0}^{m} \left[\frac{\alpha^s}{s!} \frac{\partial^s}{\partial x_3^s} \left(\sum_{q=0}^{m-s} (C_{3ikl}^{(m-s-q)} + \alpha_{,r} C_{rikl}^{(m-s-q-1)} (1 - \delta_{m0}))\, u_{k,l}^{(q)n} \right) \right] \right\} \Big|_{x_3=0} = \delta_{0m} p_i^n$$

$$\sum_{s=0}^{m} \left(\frac{\alpha^s}{s!} \frac{\partial^s}{\partial x_3^s} u_i^{(m-s)n} \right) \Big|_{x_3=0} = \delta_{0m}\chi_i^{(0)n} + \delta_{1m}\chi_i^{(1)n}$$

$$\sum_{s=0}^{m} \left(\frac{\alpha^s}{s!} \frac{\partial^s}{\partial x_3^s} C_{ijkl}^{(m-s)} \right) \Big|_{x_3=0} = \delta_{0m} C_{ijkl}^{(0)}$$

We shall subsequently assume that the characteristics of the reference medium $\{\rho, \lambda, \mu\}^{(0)}$ are known, that is, the problem will involve the refinement of the characteristics of the medium investigated. The structure of relationships (1.5) and (1.6) enables one to search successively using $m = 1, 2, \ldots$ $\{u_i^n, \rho, C_{ijkl}\}^{(m)}$. We note that the initial problem of determining

ρ, C_{ijkl}, u_i^n from N relationships of the form of (1.1)-(1.4) is non-linear (since these relationships contain the derivatives of the required functions). In this sense the transition from relationships (1.5) and (1.6) can be considered as a procedure based on the perturbation method, which is widely used in solving problems concerned with the propagation of elastic waves /6-8/.

We will now present an algorithm for finding $u_k^{(m)n}(\mathbf{x}, t)\ \{C_{ijkl}, \rho\}^{(m)}(\mathbf{x})$.

We note that the first matrix equation from (1.5) (when $m = 0$) does not contain the unknowns $\{\rho, C_{ijkl}, u_i^n\}^{(s)}$ $(s = 1, 2, \ldots)$ and, together with the first three conditions of (1.6), which are considered when $m = 0$, there is a series of ordinary initial-boundary value problems in $u_i^{(0)n}(\mathbf{x}, t)$ which describe the propagation of elastic waves in a domain Ω which is occupied by a homogeneous isotropic medium. For simplicity, we shall henceforth assume that the solutions of these problems are known and have the form $u_i^{(0)n}(\mathbf{x}, t) = \sin(a_n t)\, g_i^n(\mathbf{x})$ (there is no summation over n) which naturally imposes constraints on $\{\varphi, \psi, p, f\}_i^n$, that is, on the conditions for the initiation of the elastic processes in the medium investigated. It is noted that the actual form of $\{\varphi, \psi, p, f\}_i^n$ can be obtained by the direct substitution of $u_i^{(0)n}(\mathbf{x}, t)$ into (1.1)-(1.3) after the substitution $\{\rho, C_{ijkl}\}^{(0)} \to \{\rho, C_{ijkl}\}$.

2. In accordance with (1.5) and (1.6), which are equations in $\{\rho, C_{ijkl}, u_i^{(n)}\}^{(1)}$, we have

$$\rho^{(0)} u_i^{(1)n\cdot\cdot} - C_{ijkl}^{(0)} u_{k,\,lj}^{(1)n} = -\rho^{(1)} u_i^{(0)n\cdot\cdot} + (C_{ijkl}^{(1)} u_{k,\,l}^{(0)n})_{,\,j} \tag{2.1}$$

$$u_i^{(1)n}\big|_{t=0} = 0, \quad u_i^{(1)n\cdot}\big|_{t=0} = 0 \tag{2.2}$$

$$(C_{3jkl}^{(0)} u_{k,\,l}^{(1)n} + C_{3jkl}^{(1)} u_{k,\,l}^{(0)n} + \alpha C_{3jkl}^{(0)} u_{k,\,l3}^{(0)n} - \alpha_{,\,r} C_{rjkl}^{(0)} u_{k,\,l}^{(0)n})\big|_{x_3=0} = 0$$

$$(u_i^{(1)n} + \alpha u_{i,\,3}^{(0)n})\big|_{x_3=0} = \chi_i^{(1)n}, \quad C_{ijkl}^{(1)}\big|_{x_3=0} = 0$$

The problem of determining these functions is to a large extent analogous to the linearized diagnostic problem in /1, 9, 10/ and we shall therefore confine ourselves to the presentation of a scheme for solving it. The problem is divided into two steps: Step 1 is the determination of $u_i^{(1)n}(\mathbf{x}, t)$ and Step 2 is the recovery of $\{\rho, C_{ijkl}\}^{(1)}(\mathbf{x})$ from the right-hand side of the matrix Eq.(2.1).

Step 1. Let us apply the operator $L = (\partial_t + a_n^2 I)$ (I is a unit operator) to Eq.(2.1) after which, for each fixed n, we get

$$\rho^{(0)} v_i^{\cdot\cdot} - C_{ijkl}^{(0)} v_{k,\,lj} = 0 \tag{2.3}$$

For the unknown vector function $v_i = u^{(1)n\cdot} + a_n^2 u_i^{(1)n}$ which has been introduced (there is no summation over n), we obtain from the first three conditions of (2.2)

$$v_i\big|_{t=0} = 0, \quad v_i\big|_{x_3=0} = L\chi_i^{(1)n}, \quad v_{r,\,3}\big|_{x_3=0} = -L\chi_{3,\,r}^{(1)n}/\mu^{(0)}, \tag{2.4}$$

$$v_{3,\,3}\big|_{x_3=0} = -\lambda^{(0)} L\chi_{r,\,r}^{(1)\cdot\cdot}/(\lambda^{(0)} + 2\mu^{(0)})$$

When account is taken of (2.3), the boundary conditions (2.4) enable one to find $\{U, U_{,3}, \mathbf{W}, \mathbf{W}_{,3}\}$ when $x_3 = 0$ where $U = \operatorname{div} \mathbf{v}$, $\mathbf{W} = (W_1, W_2, W_3) = \operatorname{rot} \mathbf{v}$.

It follows from the first condition of (2.4) that the initial conditions for these functions will be homogeneous: $U|_{t=0} = 0$, $\mathbf{W}|_{t=0} = 0$. Application of the div and rot operators to Eqs.(2.3) enables one to obtain

$$\rho^{(0)} U^{\cdot\cdot} - (\lambda^{(0)} + 2\mu^{(0)})\,\Delta U = 0, \quad \rho^{(0)} \mathbf{W}^{\cdot\cdot} - \mu^{(0)} \Delta \mathbf{W} = 0$$

We therefore get a wave equation in U and each of the three components (of which only two are independent) of the vector function $\mathbf{W}$, a homogeneous initial condition and two boundary conditions when $x_3 = 0$: the boundary value of the required function and its normal derivative which constitutes a non-hyperbolic Cauchy problem (a Cauchy problem with data on a non-spatial manifold) in the case of the wave equation /11, 12/.

The problem is classically ill-posed in the class of functions $C^n(R_+^3 \times R_+)$: its solution does not exist for any values of the Cauchy data (from this class) when $x_3 = 0$. This makes the diagnostic problem ill-posed as a whole. However, the solution of a non-hyperbolic Cauchy problem is unique and its explicit representations (together with an investigation of the stability) are presented in /13, 14/. It is noted that, in the class of analytical functions, a non-hyperbolic Cauchy problem for the wave equation is classically well-posed (according to Hadamard) and can be reduced to a conventional Cauchy problem with the aid of a Volterra substitution /11/: $it^* \to x_3$, $ix_3^* \to t$ (here, i is the square root of -1) after which Kirchhoff's formula can be used to obtain the solution. By finding U, W_1 and W_2 it is possible to recover $\mathbf{v} = (v_1, v_2, v_3)$ with the aid of (2.4) and then to determine $u_i^{(1)n}(\mathbf{x}, t)$

when $t=0$ from the equations $u_i^{(1)n\cdot\cdot} + a_n{}^2 u_i^{(1)n} = v_i$ and the initial conditions (2.2). This constitutes the first step in the problem. The procedure involved in this step must be carried out N times.

Step 2. Knowing $u_i^{(1)n}(\mathbf{x}, t)$, it is possible to find the right-hand sides of Eqs.(2.1) which, in accordance with the assumption which has been adopted, will have the form $\sin(a_n t) F_i^{(1)n}(\mathbf{x})$ (there is no summation over n). The problem in the second step will therefore involve the determination of $\{\rho, C_{ijkl}\}^{(1)}(x)$ from the equations

$$a_n{}^2 \rho^{(1)} g_i{}^n + (C_{ijkl}^{(1)} g_{k,\,l}^n)_{,\,j} = F_i^{(1)n} \tag{2.5}$$

(no summation over n) and the boundary conditions (2.2)). The number N (the number of different experimental test regimes) is selected such that the number of independent scalar equations of the form of (2.5) should correspond to the number of independent unknowns.

In the case of anisotropy of a general form, the required tensor $C_{ijkl}^{(1)}(\mathbf{x})$ contains 21 independent components. If, however, it is known "a priori" that the material being investigated is isotropic, then it is sufficient to take $N=1$ when determining the independent characteristics of the medium $\{\rho, \lambda, \mu\}^{(1)}(\mathbf{x})$, that is, in this case it suffices to carry out a single dynamic trial experiment with measurement of the three components of the displacement vector on the surface of the body. Finally, a number of requirements must be satisfied regarding the functions $g_i{}^n(\mathbf{x})$, that is, regarding the conditions of the initiation of the elastic waves. In the general case the solution of the system of Eqs.(2.5) is attended by considerable difficulty. The problem is somewhat simplified with a special choice of $g_i{}^n(\mathbf{x})$ /9/ which enables one, in a number of cases, to find $\{\rho, C_{ijkl}\}^{(1)}(\mathbf{x})$ uniquely without invoking the boundary conditions for these characteristics, that is, without specifying conditions of the form of the last relationships from (2.2). This question will be clarified in greater detail when an actual example is considered.

3. In order to determine $\{\rho, C_{ijkl}, u_i{}^n\}^{(2)}$ from relationships (1.5) and (1.6), besides the latter we shall have

$$\begin{gathered}
\rho^{(0)} u_i^{(2)n\cdot\cdot} - C_{ijkl}^{(0)} u_{k,\,lj}^{(2)n} = -\rho^{(2)} u_i^{(0)n\cdot\cdot} + \\
(C_{ijkl}^{(2)} u_{k,\,l}^{(0)n})_{,\,j} + [(C_{ijkl}^{(1)} u_{k,\,l}^{(1)n})_{,\,j} - \rho^{(1)} u_i^{(1)n\cdot\cdot}] \\
u_i^{(2)n}\big|_{t=0} = 0, \quad u_i^{(2)n\cdot}\big|_{t=0} = 0
\end{gathered} \tag{3.1}$$

$$\begin{gathered}
C_{3jkl}^{(0)} u_{k,\,l}^{(2)n}\big|_{x_3=0} = -\{C_{3jkl}^{(2)} u_{k,\,l}^{(0)n} + [C_{3jkl}^{(1)} u_{k,\,l}^{(1)n} + \alpha C_{3jkl}^{(0)} u_{k,\,l3}^{(1)n} + \alpha (C_{3jkl}^{(1)} u_{k,\,l}^{(0)n})_{,\,3} + \\
{}^1/_2 \alpha^2 C_{3jkl}^{(0)} u_{k,\,l33}^{(0)n} + \alpha_{,\,r} (C_{rjkl}^{(0)} u_{l,\,l}^{(1)n} + C_{rjkl}^{(1)} u_{k,\,l}^{(0)n}) + C_{rjkl}^{(0)} \alpha (\alpha_{,\,r} u_{k,\,l}^{(0)})_{,\,3}]\}\big|_{x_3=0} \\
\{u_i^{(2)n} + [\alpha u_i^{(1)n} + {}^1/_2 \alpha^2 u_i^{(0)n}]\}\big|_{x_3=0} = 0 \\
\{C_{ijkl}^{(2)} + [\alpha C_{ijkl,\,3}^{(1)}]\}\big|_{x_3=0} = 0
\end{gathered} \tag{3.2}$$

In relationships (3.1) and (3.2), the expressions composed of functions which have already been found are separated out inside square brackets. Let us represent the required solution $u_i^{(2)n}(\mathbf{x}, t)$ in the form $u_i^{(2)n} = U_i{}^n + V_i{}^n$, where the matrix function $V_i{}^n(\mathbf{x}, t)$ satisfies relationships (3.1) and (3.2) in which only expressions within square brackets form the right-hand sides. The problem of determining $V_i{}^n$ in the mathematical scheme consists of several conventional initial-boundary value problems for the equations in the dynamic theory of elasticity which describe the propagation of elastic waves in a half space under the action of forces and force effects on the boundary. After finding $V_i{}^n(\mathbf{x}, t)$, the problem of determining $U_i{}^n$, $C_{ijkl}^{(2)}$ and $\rho^{(2)}$ from (3.1) and (3.2) is completely analogous to the problem of determining $u_i^{(1)n}$, $C_{ijkl}^{(1)}$ and $\rho^{(1)}$ from (2.1) and (2.2) considered in Sect.2.

It is seen that, in this way, it is possible to obtain successively all the terms in the expansions for the characteristics of the elastic processes $u_i{}^n(\mathbf{x}, t)$ which are used for the diagnosis and, what is more important within the framework of this problem, the characteristics of the medium being investigated $\{\rho, C_{ijkl}\}(\mathbf{x})$.

4. As an example let us consider the simplest case when the semibounded body is the half space $x_3 \geqslant 0$ $(\gamma \equiv 0)$ and it is known beforehand that the medium under investigation is isotropic and its characteristics are solely dependent on the distance to the surface. Then (under conditions for the excitation of elastic waves which are independent of x_1 and x_2), the diagnostic problem becomes one-dimensional: the characteristics of the medium are functions of x_3, while the characteristics of the processes are functions of x_3 and the time t.

The one-dimensional problem of determining the characteristics of an isotropic inhomogeneous medium or a medium with inhomogeneous initial deformations has been investigated previously /15, 16/. At the same time, it should be possible to do this without the simplifying assumptions which are necessary to solve the spatial problem. This example, therefore, does not pretend to be a new investigation of the one-dimensional problem, but solely illustrates the approach which is laid down in this paper as it is applied to the simplest case.

The number of required characteristics of the medium is reduced to three: $\{\rho, \lambda, \mu\}(x_3)$. Hence, in (1.5), (1.6) and subsequently, we put $N = 1$. When this is done, relationships (2.1) and (2.2) take the form (there is no summation over i)

$$\rho^{(0)}u_i^{(1)1\cdot\cdot} - G_i^{(0)}u_{i,33}^{(1)1} = (G_i^{(1)}u_{i,3}^{(0)1})_{,3} - \rho^{(1)}u_i^{(0)1} \tag{4.1}$$

$$G_i^{(m)} = \mu^{(m)} + \delta_{i3}(\lambda^{(m)} + \mu^{(m)})$$

$$u_i^{(1)1}\big|_{t=0} = u_i^{(1)1\cdot}\big|_{t=0} = 0;\quad u_i^{(1)1}\big|_{x_3=0} = \chi_i^{(1)1}(t),\quad u_{i,3}^{(1)1}\big|_{x_3=0} = S_i^{(1)1}(t)$$

By applying the operator $L = (\partial_t^2 + a_i^2 I)$ to these relationships and allowing for the fact that $u_i^{(0)1} = \sin(a_i t)\, g_i(x_3)$ (no summation over i), we obtain for $v_i = Lu_i^{(1)1}$

$$\rho^{(0)}v_i^{\cdot\cdot} = G_i^{(0)}v_{i,33} = 0 \tag{4.2}$$

$$v_i\big|_{t=0} = 0;\quad v_i\big|_{x_3=0} = L\chi_i^{(1)1},\quad v_{i,3}\big|_{x_3=0} = LS_i^{(1)1}$$

The problem of determining $\mathbf{v} = (v_1, v_2, v_3)$ corresponds to the problem in the first step of the diagnosis (2.3), (2.4) but, by virtue of the assumptions which have been made, it is one-dimensional and this means that, unlike the general spatial case, it is classically well-posed. Actually, the function $\mathbf{v}(x_3, t)$ can be extended onto the domain $t < 0$ in an antisymmetric manner: $\mathbf{v}(x_3, -t) = -\mathbf{v}(x_3, t)$ and, for each component v_i, we obtain a conventional Cauchy problem for a one-dimensional wave equation (apart from the relabelling $x_3 \leftrightarrow t$). For instance, $v_i(x_3, t)$ can be found using d'Alembert's formula. Next, in order to determine the right-hand side of the first relationship from (4.1), we apply the operator ∂_t to it and make use of the fact that $u_i^{(1)1\cdot}(x_3, 0) = 0$. We get

$$\rho^{(1)}g_i a_i^2 + (G_i^{(1)}g_{i,3})_{,3} = a_i^{-1}v_i^{\cdot}\big|_{t=0} = \Phi_i(x_3) \tag{4.3}$$

(where there is no summation over i). This system of ordinary differential equations is readily solved. The unknowns $\rho^{(1)}$ and $G^{(1)}_1 = G^{(1)}_2 = \mu^{(1)}$ occur in the first two equations of (4.3) (when $i = 1, 2$) and, moreover, $\rho^{(1)}$ occurs in a linear algebraic manner. By expressing $\rho^{(1)}$ from the first equation and substituting it into the second, we obtain the equation in $\mu^{(1)}$

$$A\mu^{(1)}_{,3} + B\mu^{(1)} + C = 0;\ A(x_3) = a_1g_1g_{2,33} - a_2g_2g_{1,33},\ B(x_3) = a_1g_1g_{2,3} - a_2g_2g_{1,3},\ C(x_3) = a_2g_2\Phi_1 - a_1g_1\Phi_2$$

We note that A and B depend on the type of loading while C also depends on the results of the measurements.

Let $\{A, B, C\}(x_3)$ be analytical functions. If $A \neq 0$, then

$$\mu^{(1)}(x_3) = -\exp\left(-\int_0^{x_3}\frac{B(\eta)}{A(\eta)}d\eta\right)\left[\int_0^{x_3}\frac{C(\eta)}{A(\eta)}\exp\left(\int_0^{\eta}\frac{B(\xi)}{A(\xi)}d\xi\right)d\eta + \mu(0)\right]$$

that is, the boundary value of this function has to be specified for the unique determination of $\mu^{(1)}$. If, at a certain point x_3^*, we have $A(x_3^*) = 0$ and $B(x_3^*) = 0$ then this enables one to find $\mu^{(1)}(x_3^*) = -C(x_3^*)/B(x_3^*)$ and this means that the uniqueness of what is found is achieved without the use of the boundary value $\mu^{(1)}(0)$. If $A \equiv 0$ and $B \not\equiv 0$, we have an algebraic equation in $\mu^{(1)}$ and $\mu^{(1)} = -C/B$. If $A \equiv B \equiv 0$, this means that it is impossible to determine $\mu^{(1)}$ under the given loads which excite non-stationary elastic processes in the medium under investigation.

After $\mu^{(1)}$ has been determined from the first equation of (4.3), we find

$$\rho^{(1)}(x_3) = [\Phi_1 - (\mu^{(1)}g_{1,3})_{,3}]/(a_1g_1),\ a_1g_1 \not\equiv 0$$

In order to determine $G_3^{(1)} = \lambda^{(1)} + 2\mu^{(1)}$, we substitute the result which has been obtained for $\rho^{(1)}$ into the third equation of (4.3). We obtain the equation

$$(G_3^{(1)}D)_{,3} = \Phi_3 - \rho^{(1)}a_3^2g_3,\ D = g_{3,3}$$

which is readily integrated. We note that, if $D = 0$ at just a single point x_3^* then, by virtue of the boundedness of $G_3^{(1)}$, we have

$$(G_3^{(1)}D)\big|_{x_3=x_3^*} = 0$$

and

$$G_3^{(1)}(x_3) = \frac{1}{D(x_3)} \int_{x_3^*}^{x_3} [\Phi_3(\eta) - \rho^{(1)}(\eta) g_3(\eta) a_3^2] \, d\eta$$

that is, it is not necessary to specify the boundary value of $G_3^{(1)}(0)$

It is noted that, subsequently when determining $\mu^{(m)}, \lambda^{(m)}$ $(m = 2, 3, \ )$, the coefficients A, B and D are independent of m and, in the second step of the problem, $\rho^{(m)}, \lambda^{(m)}$ $\mu^{(m)}$ are sought using analogous formulae.

We now present some results of numerical calculations. In relationships (4.1) the variables and the required functions have been made dimensionless using the formulae

$$\bar{\rho} = \frac{\rho}{\rho^{(0)}}, \quad \bar{\mu} = \frac{\mu}{E^{(0)}}, \quad \bar{\lambda} = \frac{\lambda}{E^{(0)}}, \quad \bar{t} = \frac{t}{a}\sqrt{\frac{E^{(0)}}{\rho^{(0)}}},$$

$$\bar{x}_3 = \frac{x_3}{a}, \quad u_i^{(m)} = \frac{u_i^{(m)}}{\|g_3\|_{C^1}}, \quad E^{(0)} = \lambda^{(0)} + 2\mu^{(0)}$$

For the calculations, we have used

$$a_1 = 3, a_2 = 2, a_3 = 1, \bar{g}_1 = \cos 3\bar{x}_3, \bar{g}_2 = \cos 2\bar{x}_3, \bar{g}_3 = \cos x_3, \bar{\chi}_1^{(1)1} = \chi_2^{(1)1} = 0,$$
$$\chi_3^{(1)1} = 0,05 (\sin \bar{t} - 0,5 \sin 2\bar{t})$$

As a result: $\bar{\rho} = 1, \bar{\mu} = 0.5$ (i.e. $\bar{\rho}^{(m)}, \bar{\mu}^{(m)} = 0$ when $m \geqslant 1$). The ratio $E = \lambda + 2\mu$ to $E^{(0)}$ is shown in Fig.2. The dot-dash line takes account of the first and second terms, and the solid line takes into account the first three terms of the expansion. Inclusion of the fourth term in the expansion of E in powers of the small parameter does not make any appreciable contribution to the calculation of this characteristic which is evidence of the convergence of the series.

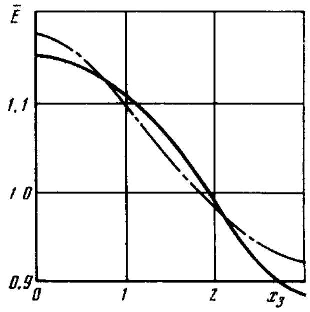

Fig.2

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THE METHOD OF PROJECTION AND DECOMPOSITION OF ANALYTICAL FUNCTIONS IN BOUNDARY-VALUE PROBLEMS OF ELASTICITY THEORY*

A.M. TSALIK

A technique is proposed for reducing boundary-value problems of elasticity theory in multiply connected regions to a system of algebraic equations. The technique is based on the projection method for analytical functions of a complex variable combined with decomposition of the original region. The starting equations are provided by the Laurent series expansion of the necessary and sufficient condition of analyticity of functions. The coordinate functions are the terms of the Laurent series for the required potentials of elasticity theory in each of the subregions obtained from the original region by decomposition. The proposed method avoids the construction of integral equations, while preserving the advantages of the boundary-element method.

1. ***Analyticity conditions.*** The necessary and sufficient condition for the function Φ to be analytic in a given region B with the boundary ∂B may be represented in the form /1/

$$\int\limits_{\partial B} \frac{\Phi(t)\,dt}{t-z} = 0, \quad z \notin B \tag{1.1}$$

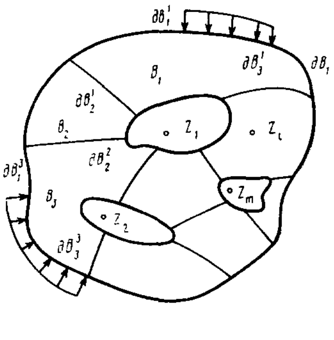

Fig.1

We assume that B is an arbitrary, closed, multiply connected region whose boundary ∂B satisfies the Hölder conditon (Fig.1), and the point at infinity does not belong to B. If $\{z_m\}$ are arbitrary points of the interior subregions that do not belong to B, then condition (1.1), after expansion in a Laurent series, can be replaced by an infinite system of equations for the analytical function Φ,

$$\int\limits_{\partial B} \Phi(t)\,\xi\,dt = 0 \quad (\xi = t^k, (t-z_m)^{-k-1}, k = 0, 1, \ldots) \tag{1.2}$$

Assume that the given region $B = \cup B^i$ is decomposed so that inside each subregion B^i the function $\Phi(z)$ is representable by its Laurent series

$$\Phi^i = \sum_{m=1}^{M} \sum_{s=-1}^{-\infty} \Phi_{ms}^i (z-z_m)^s + \sum_{s=0}^{\infty} \Phi_s^i (z-z_i)^s \tag{1.3}$$

where $M+1$ is the connectivity of the region B, z_i is an arbitrary point of the subregion B^i. Analytical continuity conditions for Φ should be satisfied on the joining curves of the subregions.

The functions in the expansion (1.3) are selected as the coordinate functions of the

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